TEMPERATURE STRESSES IN AN ANNULAR PLATE MADE OF REINFORCED LAMELLAR MATERIAL

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Solutions are obtained for a nonsteady-state problem of heat conduction and of the corresponding quasistatic thermoelasticity problem for an annulus made of a reinforced lamellar material.

We consider a plate in the form of an annulus. We assume that it possesses thermal and elastic cylindrical anisotropy and that it has a plane of symmetry (the z axis is normal to this plane).

We assume that heat transfer with the exterior medium through the surface of the annulus obeys Newton's law. The temperature of the medium washing the surfaces $z = \pm\delta$ is an arbitrary function of time $t_c(\tau)$, while the temperatures of the medium washing the concentric surfaces $\rho = r$, $\rho = R$ are $t_r(\tau)$, $t_R(\tau)$.

For the determination of the nonsteady-state temperature field in such a plate we have the heat conduction equation and the boundary conditions [1]

$$\frac{\partial^2 t}{\partial \rho^2} + \frac{1}{\rho} \cdot \frac{\partial t}{\partial \rho} - \mathrm{Bi}_z (t - t_c) = \frac{\partial t}{\partial \mathrm{Fo}}, \tag{1}$$

$$\left.\begin{aligned} \frac{\partial t}{\partial \rho} &= \mathrm{Bi}_r (t - t_r) \quad \text{for} \quad \rho = r \\ \frac{\partial t}{\partial \rho} &= -\mathrm{Bi}_R (t - t_R) \quad \text{for} \quad \rho = R \end{aligned}\right\}, \tag{2}$$

$$t = 0 \quad \text{for} \quad \tau = 0. \tag{3}$$

Making use of the Laplace transform with respect to the variable τ, we seek the solution of the boundary value problem (1)-(3) in the form

$$\begin{aligned} t = \int\limits_0^{\mathrm{Fo}} \sum_{n=1}^{\infty} \Bigg\{ & t_r(\tau)\,\mathrm{Bi}_r \left[\frac{\nu_n}{r} Z_0^{(1)}\left(\frac{\nu_n}{r}\rho,\ m\nu_n\right) - \mathrm{Bi}_R Z_0^{(0)}\left(\frac{\nu_n}{r}\rho,\ m\nu_n\right)\right] + t_R(\tau)\,\mathrm{Bi}_R \left[\frac{\nu_n}{r} Z_0^{(1)}\left(\frac{\nu_n}{r}\rho,\ \nu_n\right)\right. \\ & \left. + \mathrm{Bi}_r Z_0^{(0)}\left(\frac{\nu_n}{r}\rho,\ \nu_n\right)\right] + \frac{\mathrm{Bi}_z r}{\nu_n} t_c(\tau) \left[\mathrm{Bi}_r Z_0^{(1)}\left(\frac{\nu_n}{r}\rho,\ m\nu_n\right) + \mathrm{Bi}_R Z_0^{(1)}\left(\frac{\nu_n}{r}\rho,\ \nu_n\right)\right. \\ & \left. + \frac{\mathrm{Bi}_r \mathrm{Bi}_R}{\nu_n} \tau \left(Z_0^{(0)}\left(\frac{\nu_n}{r}\rho,\ \nu_n\right) - Z_0^{(0)}\left(\frac{\nu_n}{r}\rho,\ m\nu_n\right)\right)\right]\Bigg\} \frac{e^{s_n(\mathrm{Fo}-\tau)}}{\Phi(\nu_n)}\, d\tau + \mathrm{Bi}_z \int\limits_0^{\mathrm{Fo}} t_c(\tau)\, e^{-\mathrm{Bi}_z(\mathrm{Fo}-\tau)}\, d\tau, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \Phi(\nu_n) = -\frac{r^2}{2\nu_n} \Bigg[& -Z_0^{(1)}(m\nu_n,\ \nu_n)\left(\frac{\mathrm{Bi}_R}{r} + \mathrm{Bi}_r \mathrm{Bi}_R + \frac{\nu_n^2 m}{r^2}\right) + Z_0^{(1)}(\nu_n,\ m\nu_n)\left(m\,\mathrm{Bi}_r \mathrm{Bi}_R - \frac{\mathrm{Bi}_r}{r} + \frac{\nu_n^2}{r^2}\right) \\ & + \frac{\nu_n}{r} Z_1^{(1)}(m\nu_n,\ \nu_n)(\mathrm{Bi}_r - m\,\mathrm{Bi}_R) - \frac{\nu_n}{r} Z_0^{(0)}(m\nu_n,\ \nu_n)(m\,\mathrm{Bi}_r - \mathrm{Bi}_R) + \frac{2\nu_n}{r^2} Z_1^{(1)}(m\nu_n,\ \nu_n)\Bigg]; \end{aligned}$$

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$$Z_i^{(j)}(x,\ y)=J_i(x)Y_j(y)-Y_i(x)J_j(y)\quad (i,\ j=0;\ 1);$$

$$m=\frac{R}{r};\quad s_n=-\left(\mathrm{Bi}_z+\frac{\nu_n^2}{r^2}\right).$$

For the case of an isotropic plate, when on the cylindrical surfaces we have conditions of the first kind ($\alpha_r=\infty$, $\alpha_R=\infty$), this solution is given in [2]. But if the interior surface of the ring is insulated and on the exterior surface we have a condition of the first kind, then it is easy to obtain from (4) the known [3] solution of the problem by making $\alpha_r\to 0$, $\alpha_R\to\infty$ and taking $t_c(\tau)=0$.

Taking in (4) the limit as $\alpha_r\to 0$, $\alpha_R\to 0$, we obtain the following expression for the temperature field:

$$t=\mathrm{Bi}_z\int_0^{\mathrm{Fo}} t_c(\tau)\,e^{-\mathrm{Bi}_z(\mathrm{Fo}-\tau)}\,d\tau. \tag{5}$$

For the case when the temperature of the exterior medium varies at the initial moment by some quantity t_0, considered constant in what follows, this temperature field becomes

$$t=t_0\left(1-e^{-\mathrm{Bi}_z\mathrm{Fo}}\right). \tag{6}$$

Let us determine the temperature stresses induced in the ring by the temperature field (4) obtained, by making use of the necessary [4] relations of the thermoelasticity for a plate:

$$\begin{aligned}\sigma_\rho&=E_*\left(\frac{\partial u}{\partial\rho}+\nu_{\rho\varphi}\frac{u}{\rho}-\alpha_t^* t\right);\\ \sigma_\varphi&=E_*\left(k^2\frac{u}{\rho}+\nu_{\rho\varphi}\frac{\partial u}{\partial\rho}-\alpha_*^t t\right),\end{aligned} \tag{7}$$

where u satisfies the equation

$$\begin{aligned}&\frac{\partial^2 u}{\partial\rho^2}+\frac{1}{\rho}\cdot\frac{\partial u}{\partial\rho}-k^2\frac{u}{\rho^2}=f(\rho,\ \mathrm{Fo});\\ &f(\rho,\ \mathrm{Fo})=\alpha_t^*\frac{\partial t}{\partial\rho}+(\alpha_t^*-\alpha_*^t)\frac{t}{\rho}.\end{aligned} \tag{8}$$

The solution of the equation (8) can be written in the form

$$u=\left(c_1+\frac{\Psi_-(\rho)}{2k}\right)\rho^k+\left(c_2-\frac{\Psi_+(\rho)}{2k}\right)\rho^{-k}, \tag{9}$$

where $\Psi_\pm(\rho)=\int_r^\rho \rho^{\pm k+1}f(\rho,\ \mathrm{Fo})d\rho$, while the integration constants c_1 and c_2 are subject to determination from the conditions that the plate is free of external load: $\sigma_\rho=0$ for $\rho=r$, $\rho=R$.

Determining the integration constants after the computation of the necessary integrals [5], we obtain the following expressions for the components of the temperature stresses:

$$\sigma_\rho=\frac{E_*}{2}\left[2\left(\frac{\rho}{r}\right)^{k-1}\frac{1-\left(\frac{R}{\rho}\right)^{2k}}{1-m^{2k}}\alpha_t^*t|_{\rho=r}\right.$$

$$\left.+\left(\frac{R}{\rho}\right)^{k+1}\frac{1-\left(\frac{\rho}{r}\right)^{2k}}{1-m^{2k}}\chi(R)+(k\nu_{\varphi\rho}+1)\rho^{k-1}\Psi_-(\rho)+(1-k\nu_{\varphi\rho})\rho^{-k-1}\Psi_+(\rho)-2\alpha_t^*t\right],$$

$$\sigma_\varphi=\frac{E_*}{2}\left\{\frac{2k}{1-m^{2k}}\left(\frac{\rho}{r}\right)^{k-1}\left[\left(\frac{R}{\rho}\right)^{2k}+1\right]\alpha_t^*t|_{\rho=r}\right.$$

$$\left.-\left(\frac{R}{\rho}\right)^{k+1}\frac{k}{1-m^{2k}}\left[\left(\frac{\rho}{r}\right)^{2k}+1\right]\chi(R)+(\nu_{\rho\varphi}+k)\rho^{k-1}\Psi_-(\rho)+(\nu_{\rho\varphi}-k)\rho^{-k-1}\Psi_+(\rho)-2\alpha_*^t t\right\}, \tag{10}$$

where

$$\chi(R)=2\alpha_t^* t|_{\rho=R} - (k\nu_{\varphi\rho}+1)R^{k-1}\Psi_-(R) + (k\nu_{\varphi\rho}-1)R^{-k-1}\Psi_+(R);$$

$$\Psi_\pm(\rho)=\alpha_t^*(\rho^{\pm k+1}t - r^{\pm k+1}t|_{\rho=r}) + (\nu_{\rho\varphi}+k)(\alpha_\rho^t \pm k\alpha_\varphi^t)$$

$$\times\int_0^{\mathrm{Fo}}\sum_{n=1}^{\infty}\frac{r^{\pm k+1}}{\nu_n^{\pm k}}\Bigg\{\Bigg[\mathrm{Bi}_r t_r(\tau)\left(\frac{\nu_n}{r}J_1(m\nu_n)+\mathrm{Bi}_R J_0(m\nu_n)\right)$$

$$+\mathrm{Bi}_R t_R(\tau)\left(\frac{\nu_n}{r}J_1(\nu_n)+\mathrm{Bi}_r J_0(\nu_n)\right)+\frac{\mathrm{Bi}_z r}{\nu_n}t_c(\tau)P_J\Bigg]$$

$$\times\left[L_Y(\pm k,\ 0;\ 1)+(\pm k-1)L_Y(\pm k-1,\ -1;\ 0)\right]-\Bigg[\mathrm{Bi}_r t_r(\tau)\left(\frac{\nu_n}{r}Y_1(m\nu_n)-\mathrm{Bi}_R Y_0(m\nu_n)\right)$$

$$+\mathrm{Bi}_R t_R(\tau)\left(\frac{\nu_n}{r}Y_1(\nu_n)+\mathrm{Bi}_r Y_0(\nu_n)\right)+\frac{\mathrm{Bi}_z r}{\nu_n}t_c(\tau)P_Y\Bigg]$$

$$\times\left[L_J(\pm k,\ 0;\ 1)-(1\mp k)L_J(\pm k-1,\ -1;\ 0)\right]\Bigg\}\frac{e^{s_n(\mathrm{Fo}-\tau)}}{\Phi(\nu_n)}d\tau$$

$$-\frac{(\nu_{\rho\varphi}\pm k)(\alpha_\rho^t\pm k\alpha_\varphi^t)}{(\pm k+1)}(\rho^{\pm k+1}-r^{\pm k+1})\mathrm{Bi}_z\int_0^{\mathrm{Fo}}t_c(\tau)e^{-\mathrm{Bi}_z(\mathrm{Fo}-\tau)}d\tau; \tag{11}$$

$$P_G=\mathrm{Bi}_r G_1(m\nu_n)+\mathrm{Bi}_R G_1(\nu_n)+\frac{\mathrm{Bi}_r\mathrm{Bi}_R r}{\nu_n}\left(G_0(\nu_n)-G_0(m\nu_n)\right);$$

$$L_G(\pm k-i,\ -i;\ j)=\frac{\rho'}{r}S_{\pm k-i,\,-i}\left(\frac{\nu_n}{r}\rho\right)G_j\left(\frac{\nu_n}{r}\rho\right)$$

$$-S_{\pm k-i,\,-i}(\nu_n)G_j(\nu_n)\quad (G=J;\ Y),\quad (i,\ j=0;\ 1).$$

If in the formulas (10) the temperature field is stationary and $E_\varphi = E_\rho = E$, $\alpha_\rho^t = \alpha_\varphi^t = \alpha_t$, $\nu_{\rho\varphi} = \nu_{\varphi\rho} = \nu$, then we arrive at the known [6-8] expressions of the temperature stresses for an isotropic plate under steady-state heat conditions.

The largest temperature stresses induced by the temperature field (6) have the form

$$\sigma_\rho^* = \frac{\sigma_\rho\cdot 10^{-5}}{t_0} = E_* M\cdot 10^{-5}\left\{\frac{m^{\frac{k^2-1}{2k}}}{1-m^{2k}}(1-m^{k+1})^{\frac{k+1}{2k}}(1-m^{k-1})^{\frac{k-1}{2k}}\left[\left(\frac{k+1}{k-1}\right)^{\frac{k-1}{2k}}+\left(\frac{k+1}{k-1}\right)^{-\frac{k+1}{2k}}\right]-1\right\}, \tag{12}$$

$$\sigma_\varphi^* = \frac{\sigma_\varphi|_{\rho=r}\cdot 10^{-5}}{t_0} = E_* M\cdot 10^{-5}\left(\frac{1-2m^{k+1}+m^{2k}}{1-m^{2k}}k-1\right),$$

where $M=\left[\frac{1+\nu_{\rho\varphi}}{k^2-1}(\alpha_t^*-\alpha_*^t)+\alpha_t^*\right](1-e^{-\mathrm{Bi}_z\mathrm{Fo}})$.

We note that the nonsteady-state temperature field, constant along the entire domain of its plate does not induce temperature stresses in the isotropic plate. In an anisotropic plate, temperature stresses arise and, as seen from (12), they depend on the heat transfer with its lateral surfaces, the nonsteady-state temperature field and the polar radius.

Let us assume that the annular plate is made of epoxy resin, reinforced with equal-strength glass bands. The necessary physicomechanical characteristics for such a plate are given in [4, 9]. In order to elucidate the effect of the heat transfer with the lateral surfaces $z = \pm\delta$ of the plate and the nonsteadiness of the temperature field on the temperature stresses in the reinforced plates, the stresses (12) have been computed for a reinforcing coefficient $\psi' = 0.8$.

In Fig. 1 diagrams of the variations of these stresses as a function of the Fourier numbers are given for several values of the Biot numbers. It is clear from the diagrams that the largest radial and annular temperature stresses are always contractive and are attained for a stationary heat condition. With the decrease of the heat transfer the temperature stresses will decrease.

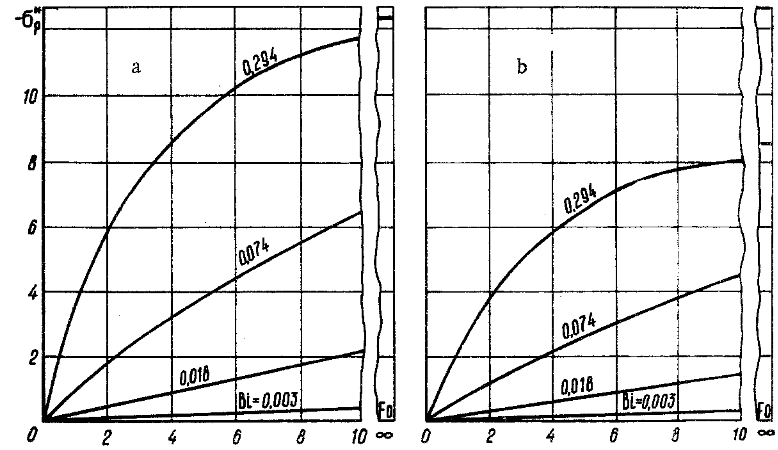

Fig. 1. The variation in time of the maximal radial (a) and annular (b) temperature stresses in an annular plate for different values of the Biot number. In (b), along the ordinates we have σ_φ^*.

NOTATION

$t(\rho,\ \tau)$	is the temperature of the plate;
τ	is the time;
$\rho,\ r,\ R$	are the polar radius, inner and outer radii of the ring related to the semithickness δ of the plate;
$Bi_z = \alpha_z\delta/\lambda_\rho$, $Bi_r = \alpha_r\delta/\lambda_\rho$, $Bi_R = \alpha_R\delta/\lambda_\rho$	are the Biot numbers on the surfaces;
$z = \pm\delta,\ \rho = r,\ \rho = R;\ \alpha_z,\ \alpha_r,\ \alpha_R$	are the heat transfer coefficients with these surfaces;
$Fo = \lambda_\rho\tau/C\delta^2$	is the Fourier number;
λ_ρ	is the heat conductivity coefficient in the direction ρ;
C	is the heat capacity;
$E_* = E_\rho(1-\nu_{\rho\varphi}\nu_{\varphi\rho})^{-1};\ k^2 = E_\varphi/E_\rho;\ \alpha_t^* = \alpha_\rho^t + \nu_{\rho\varphi}\alpha_\varphi^t;\ \alpha_*^t = \alpha_\rho^t\nu_{\rho\varphi} + k^2\alpha_\varphi^t;\ E_\rho,\ E_\varphi$	are the Young's moduli in the radial and tangential directions;
$\nu_{\varphi\rho},\ \nu_{\rho\varphi}$	are the Poisson ratios for these directions;
$\alpha_\rho^t,\ \alpha_\varphi^t$	are the linear expansion coefficients in the directions $\rho,\ \varphi$;
u	is the radial displacement;
$I_0(x),\ K_0(x)$	are the zero-order Bessel functions of the first and the second kind with imaginary argument;
$J_n(x),\ Y_n(x)$	are the Bessel functions of the first and second kind with real argument;
$S_{p,q}(x)$	is the Lommel function with real argument.

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